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Max algebra and the linear assignment problem

Dedicated to Professor Egon Balas on the occasion of his 80th birthday.

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Abstract. Max-algebra, where the classical arithmetic operations of addition and multiplication are replaced by $a \oplus b := \max(a, b)$ and $a \otimes b := a + b$ offers an attractive way for modelling discrete event systems and optimization problems in production and transportation. Moreover, it shows a strong similarity to classical linear algebra: for instance, it allows a consideration of linear equation systems and the eigenvalue problem. The max-algebraic permanent of a matrix A corresponds to the maximum value of the classical linear assignment problem with cost matrix A. The analogue of van der Waerden's conjecture in max-algebra is proved. Moreover the role of the linear assignment problem in max-algebra is elaborated, in particular with respect to the uniqueness of solutions of linear equation systems, regularity of matrices and the minimal-dimensional realisation of discrete event systems. Further, the eigenvalue problem in max-algebra is discussed. It is intimately related to the best principal submatrix problem which is finally investigated: Given an integer k, $1 \leq k \leq n$, find a $(k \times k)$ principal submatrix of the given $(n \times n)$ matrix which yields among all principal submatrices of the same size the maximum (minimum) value for an assignment. For $k = 1, 2, ..., n$, the maximum assignment problem values of the principal $(k \times k)$ submatrices are the coefficients of the max-algebraic characteristic polynomial of the matrix for A. This problem can be used to model job rotations.

Key words. max-algebra – assignment problem – permanent – regular matrix – discrete event system – characteristic maxpolynomial – best principal submatrix assignment problem – job rotation problem

1. Introduction

In the max-algebra the conventional arithmetic operations of addition and multiplication in the real numbers are replaced by

$$a \oplus b := \max(a, b), \tag{1}$$

$$a \otimes b := a + b, \tag{2}$$

where $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$. The algebraic system $(\overline{\mathbb{R}}, \oplus, \otimes)$ offers an adequate language to describe problems from communication networks (Shimbel [23]), synchronization of production (Cuninghame-Green [8]) and transportation, shortest paths (e.g. Peteanu [21], Carré [7], Gondran [17]) and discrete event systems, to mention just a few

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applications in the field of operations research and optimization. But the max-algebra is also interesting from a mathematical point of view: though the operation "subtract" does not exist within this system, nevertheless many notions from conventional linear algebra, like equation systems, eigenvalues, projections, subspaces, singular value decomposition and a duality theory can be developed within this system. The first comprehensive account on algebraic properties of max-algebra and its applications can be found in the lecture notes of Cuninghame-Green [10]. Further important surveys in this field are e.g. the monographs of Baccelli, Cohen, Olsder and Quadrat [1] and Cuninghame-Green [12], the theses of Gaubert [15] and De Schutter [14], the survey paper of Gondran and Minoux [19] and the monograph of Zimmermann [25] on optimization in ordered algebraic structures.

It turns out that there are various connections between max-algebra and the classical linear assignment problem. One such connection is that the max-algebraic analogue of the permanent of a matrix A corresponds to the maximum value of the linear assignment problem with cost matrix A. Secondly, the unique solvability of a linear system of equations in max-algebra is closely related to the solution set of a linear assignment problem. This will lead us to consider *strongly regular* and *regular* matrices which – in turn – play a crucial role in the minimal-dimensional realisation problem of discrete event dynamical systems. Since the matrices arising in this context are symmetric, it is interesting to note that a symmetric normal form can always be achieved for symmetric matrices.

In connection with the eigenvalue problem, Cuninghame-Green [11] introduced the max-algebraic analogue of the characteristic polynomial of a matrix A as *characteristic maxpolynomial*. The coefficients of this characteristic maxpolynomial are the maximum assignment problem values of principal submatrices of A. This leads to the *Best Principal Submatrix Assignment Problem (BPSAP):* Given an integer k, $1 \leq k \leq n$, find a $(k \times k)$ principal submatrix of the given matrix A which yields among all principal submatrices of the same size the maximum (minimum) value for an assignment. For certain integers k, (BPSAP) can be solved in polynomial time. In the general case the complexity status of the best principal submatrix assignment problem is still unknown to the best knowledge of the authors. There exists, however, a randomized polynomial algorithm for this problem (see Section 6), provided the entries of matrix A are polynomially bounded.

The paper is organised as follows. In the next section we introduce the basic notions of max-algebra and provide an example from transportation. Then we discuss linear equation systems in max-algebra and introduce strongly regular and regular matrices. These play a role in the next section during the discussion on a minimal-dimensional realisation of discrete event dynamical systems. It leads to assignment problem instances with symmetric cost matrices. Next the eigenvalue problem in max-algebra and its relation to the best principal submatrix problem is outlined. Finally we discuss the solution of the best principal submatrix assignment problem.

2. Max-algebra

Let us denote $a \oplus b := \max(a, b)$ and $a \otimes b := a + b$ for $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$. The operations $\oplus$ and $\otimes$ can be extended to vectors and matrices in the same way as in conventional linear algebra. Thus, if $A = (a_{ij}),\ B = (b_{ij})$ and $C = (c_{ij})$ are matrices with elements from $\overline{\mathbb{R}}$ of compatible sizes, we can write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k(a_{ik} + b_{kj})$ for all i, j. In max-algebra the unit matrix I is a square matrix of appropriate size whose diagonal elements are all 0 and whose off-diagonal elements are $-\infty$.

Let us give an example which leads to a linear system of equations in max-algebra:

Example 1. Two airplanes P_1 and P_2 arrive at the airport C after a flying time of 1 and 3 hours, respectively. Passengers may change from P_1 to P_2 within 30 minutes. Passengers changing from P_2 to P_1 need 60 minutes due to customs clearance. The planes should continue their journey at specified times b_1 and b_2. When are they to depart latest from their origin airports?

In order to model this situation we introduce the variables x_1 and x_2 which are the departing times from the original airports, and b_1 and b_2 which are the departing times from airport C. We get

$$b_1 = \max(x_1 + 60, x_2 + 180 + 60),$$
$$b_2 = \max(x_1 + 60 + 30, x_2 + 180).$$

This system can be written as a linear system of equations

$$A \otimes x = b \tag{3}$$

in max-algebra using the matrix

$$A := \begin{pmatrix} 60 & 240 \\ 90 & 180 \end{pmatrix}$$

and the vector $x := (x_1, x_2)^t$. □

We shall deal in the next section with the question under which conditions such a system of linear equations in max-algebra is uniquely solvable. This will lead us to strongly regular and regular matrices which play a role in describing discrete event-driven systems, like the following cyclic scheduling problem.

Example 2. Two suburban trains T_1 and T_2 run hourly from their home stations A_1 and A_2, respectively, on circular lines which meet in a station S. The trains start their travel in A_1 and A_2, at the times $x_1(0)$ and $x_2(0)$, respectively. In order to allow people to change trains in station S, they must wait 5 minutes after the arrival of the other train, before they can continue. One train needs 40 minutes from A_1 to S, the other train needs 45 minutes from A_2 to S. Let $x_1(r)$ and $x_2(r)$ denote the earliest departure times for train T_1 and train T_2, respectively, from their home stations. Then their schedule develops according to the following system

$$x_1(r) = \max(x_1(r-1) + 40, x_2(r-1) + 45 + 5) + 40,$$
$$x_2(r) = \max(x_1(r-1) + 40 + 5, x_2(r-1) + 45) + 45.$$

This system can be written in max-algebra as a linear recurrence

$$x(r) = A \otimes x(r-1) \otimes b \tag{4}$$

with

$$A := \begin{pmatrix} 40 & 50 \\ 45 & 45 \end{pmatrix}.$$

and $b = (40, 45)^t$. □

An interesting operational question in controlling such systems is: *How must the system be set in motion to ensure that it moves forward in regular steps?* This question asks for a constant λ such that

$$x(r+1) = \lambda \otimes x(r).$$

In other words, λ is a max-algebraic eigenvalue for which

$$A \otimes x(r) = \lambda \otimes x(r) \tag{5}$$

holds. The eigenvalue problem will be discussed in Section 5.

Now, why does the assignment problem play an important role in max-algebra? One reason is that the max-algebraic permanent of a matrix A leads to the problem of finding a permutation φ which yields a maximum value for the assignment problem with cost matrix A.

The *max-algebraic permanent* of an $n \times n$ matrix $A = (a_{ij})$ is defined in an analogue way to classical linear algebra by

$$\text{maper}(A) := \sum_{\varphi \in \mathcal{S}_n}^{\oplus} \prod_{1 \le i \le n}^{\otimes} a_{i\varphi(i)}, \tag{6}$$

where $\mathcal{S}_n$ denotes the set of all permutations of the set $\{1, 2, ..., n\}$. In conventional notation,

$$\text{maper}(A) = \max_{\varphi \in \mathcal{S}_n} \sum_{1 \le i \le n} a_{i\varphi(i)}$$

which is the maximum value of the assignment problem with cost matrix A. Throughout the paper we shall denote the set of all permutations which yield the maximum of the assignment problem with cost matrix A by $ap(A)$.

A first result concerns the max-algebraic version of the *van der Waerden Conjecture*. An $n \times n$ matrix $A = (a_{ij})$ is called *doubly stochastic*, if all $a_{ij} \ge 0$ and all row and column sums of A equal 1.

Proposition 2.1. *(Max-algebraic van der Waerden Conjecture)*
Among all doubly stochastic $(n \times n)$ matrices the max-algebraic permanent obtains its minimum for the matrix $A^* = (a_{ij})$ with $a_{ij} := 1/n$ for all i and j.

Proof. We have $\text{maper}(A^*) = \max_{\varphi \in \mathcal{S}_n} \sum_{1 \le i \le n} a^*_{i\varphi(i)} = 1$. Assume that there is a doubly stochastic matrix $X = (x_{ij})$ with $\max_{\varphi} \sum_{1 \le i \le n} x_{i\varphi(i)} < 1$. Then we get for all permutations φ: $\sum_{1\le i\le n} x_{i\varphi(i)} < 1$. This holds in particular for the permutations φ_k which map i to $i+k$ modulo n for $i = 1, 2, ..., n$. Thus we get

$$n = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} = \sum_{k=0}^{n-1}\sum_{i=1}^{n} x_{i\varphi_k(i)} < n,$$

a contradiction. Thus matrix A^* yields the minimum for the max algebraic permanent. $\square$

Later we will also deal with polynomials in max-algebra. Let x be a variable ranging in $\overline{\mathbb{R}}$. For any positive integer r we define $x^{(r)} := x \otimes x \otimes ... \otimes x$ (r-fold) and $x^{(0)} := 0$. Let coefficients $a_0, a_1, ..., a_n \in \overline{\mathbb{R}}$ be given. A *maxpolynomial* has the form

$$p(x) = \sum_{0\le r\le n}^{\oplus} a_r \otimes x^{(r)} = \max_{1\le r\le n} (a_r + rx). \tag{7}$$

This shows that the graph of maxpolynomials is a piecewise linear, convex function. Maxpolynomials admit a linear factorisation in a way similar to that described in the fundamental theorem of algebra.

Proposition 2.2. *Cuninghame-Green and Meier [13]*
Any maxpolynomial (7) can be written as a product of n linear factors

$$p(x) = a_n \otimes l_1(x) \otimes l_2(x) \otimes ... \otimes l_n(x), \tag{8}$$

where every linear factor $l_r(x)$ has either the form x or $(x \oplus b_r)$ for some $b_r \in \mathbb{R}$.

Written in a conventional way, the factorisation theorem states that any maxpolynomial can be written in the form

$$a_n + \max(x, b_1) + \max(x, b_2) + ... + \max(x, b_n).$$

The constants b_r, $1 \le r \le n$, are called the *corners of $p(x)$*.

3. Equation systems

Let an $m \times n$ matrix A with elements from $\mathbb{R}$ and a real vector $b \in \mathbb{R}$ be given. We consider the linear equation system

$$A \otimes x = b. \tag{9}$$

As in conventional linear algebra such a system may have none, exactly one or infinitely many solutions. Of special interest is the case when such a system has just *one solution*. In this respect it has been shown:

Proposition 3.1. *(Cuninghame-Green [10], Butkovič [3])*
Let A be a real $n \times n$ matrix and let $b \in \mathbb{R}^n$. The linear equation system $A \otimes x = b$ has a unique solution $\bar{x}$ if and only if the matrix $C = (c_{ij})$ with $c_{ij} := a_{ij} - b_i$ has exactly one column maximum in every row and column. The solution $\bar{x} = (\overline{x}_1, \overline{x}_2, ..., \overline{x}_n)^t$ is given by

$$\overline{x}_j := -\max_{1\le i\le n} (a_{ij} - b_i).$$

This theorem suggests that assignments play a role with respect to the unique solubility of a linear equation system. And indeed, if a linear equation system with a real $n \times n$ coefficient matrix is soluble and the solution is unique, then the linear assignment problem with cost matrix A has only one solution φ with maximum value, i.e. $|ap(A)| = 1$. An $(n \times n)$ matrix A is called *strongly regular*, if there exists a vector $b \in \mathbb{R}^n$ such that $A \otimes x = b$ has a unique solution.

Proposition 3.2. *(Butkovič [5])*
$A \in \mathbb{R}^{n\times n}$ is strongly regular if and only if there is only one solution for the linear assignment problem with cost matrix A yielding a maximum value.

In order to check whether $|ap(A)| = 1$ or not, one can proceed as follows. We say the $n \times n$ matrix $B \leq 0$ is a *normal form* of the matrix A if there is a constant z such that for all permutations

$$\sum_{i=1}^{n} a_{i\varphi(i)} = z + \sum_{i=1}^{n} b_{i\varphi(i)} \tag{10}$$

and there exists a permutation φ_0 with

$$\sum_{i=1}^{n} b_{i\varphi_0(i)} = 0.$$

It is straightforward to see that $ap(A) = ap(B)$. Matrix B can be obtained from A e.g. by applying the Hungarian Method. By permuting the rows and columns of B we can achieve that the identical permutation id lies in $ap(B)$:

$$id \in ap(B). \tag{11}$$

Assuming (11) we define a digraph $G_B(V, E)$ with the node set $V := \{1, 2, ..., n\}$ and arcs $(i, j) \in E$ if and only if $i \neq j$ and $b_{ij} = 0$. Now, $|ap(A)| = 1$, if and only if the graph G_B is acyclic. Conversely, the question whether a digraph G_B is acyclic can straightforwardly be formulated as the strong regularity question for a suitably defined square matrix.

The statement of Proposition 3.2 does not hold in non-dense subgroups of the additive group of reals, e.g. for integer matrices, but a necessary and sufficient condition for matrices over an arbitrary linearly ordered commutative group has been proved in [5].

Now we define linearly dependent vectors in max-algebra. The vectors $a_1, a_2, ..., a_n$ are called *linearly dependent*, if there are real numbers λ_j and two non-empty, disjoint subsets S and T of the set $\{1, 2, ..., n\}$ such that

$$\sum_{j\in S}^{\oplus} \lambda_j \otimes a_j = \sum_{j\in T}^{\oplus} \lambda_j \otimes a_j.$$

If such a representation is not possible, the vectors are called *linearly independent*. A square matrix is called *regular*, if its column vectors are linearly independent. Gondran and Minoux [18] showed the following criterion for the regularity of a matrix.

Proposition 3.3. *[19, 18]*
A matrix $A \in \mathbb{R}^{n\times n}$ is regular if and only if every permutation yielding the maximum value for the linear assignment problem with cost matrix A has the same parity.

Proposition 3.2 immediately implies that a strongly regular matrix is regular. Butkovič [4] showed that the question whether a matrix is regular or not is closely related to the even cycle problem.

Theorem 3.4. *[4]*
Let B be a normal form of $A \in \mathbb{R}^{n\times n}$ in which (if necessary) the rows or columns have been permuted so that $id \in ap(B)$. Then A is regular if and only if the digraph G_B does not contain an even cycle.

Since the even cycle problem has been shown to be polynomially solvable (see Robertson, Seymour and Thomas [22]), the problem to check whether a given square matrix is regular can be solved in polynomial time.

4. Minimal-dimensional realisation of discrete event systems

A classical problem is the following. An unknown system emits a sequence of real-number signals

$$G = \{g_j\}_{j=0}^{\infty}$$

at discrete time intervals. Find a compact description of the system given only this observed sequence. Solutions of this problem depend on the assumption we set on the underlying process. In system theory significant effort has been devoted to the case when the process is describable through the state vector $x(j) \in \mathbb{R}^n$ of the system at time $j = 0, 1, ...$ and the change of state is described through a linear transformation. In some synchronous processes this would become a max-algebraic linear transformation of the form

$$x(j) \mapsto x(j+1) = A \otimes x(j), \quad x(0) = b,$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and the states are observed through an *observation vector* $c \in \mathbb{R}^n$ that is

$$g_j = c^t \otimes x(j), \quad j = 0, 1, ...$$

We call g_j *Markov parameters* and a triple (A, b, c), where $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, a *realisation* of the discrete-event system (DES) emitting G if

$$g_j = c^t \otimes A^{(j)} \otimes b \quad (j = 0, 1, ...)$$

where by convention $A^{(0)} \otimes b = b$. In general, there are many trivial realisations but for a compact, economical description we seek a realisation of the least possible dimension n – a *minimal dimensional realisation* (MDR).

An important upper bound of the dimension of MDR is based on the following max-algebraic version of the Cayley-Hamilton theorem. Straubing [24] showed that for every matrix $A \in \mathbb{R}^{n\times n}$ there exists a *characteristic equation* of the form

$$\sum_{j\in S}^{\oplus} \alpha_j \otimes \lambda^{(j)} = \sum_{j\in T}^{\oplus} \alpha_j \otimes \lambda^{(j)}, \tag{12}$$

where S and T are non-empty, disjoint subsets of the set $\{0, 1, ..., n\}$ and the real coefficients α_j, $j \in S \cup T$, are determined by the entries of the matrix A.

Proposition 4.1. *(Cayley-Hamilton Theorem, Straubing [24])*
Matrix $A \in \mathbb{R}^{n\times n}$ satisfies its characteristic equation (12) when substituted for λ.

Given a sequence $G = \{g_j\}_{j=0}^{\infty}$, we define the *Hankel matrix* H_r $(r = 0, 1, 2, ...)$ associated with the DES by

$$\begin{pmatrix} g_0 & g_1 & \cdots & g_r \\ g_1 & g_2 & \cdots & g_{r+1} \\ \vdots & \vdots & & \vdots \\ g_r & g_{r+1} & \cdots & g_{2r} \end{pmatrix}.$$

Suppose that G has a realisation of order n. Thus

$$g_j = c^t \otimes A^{(j)} \otimes b \quad (j = 0, 1, ...)$$

for some $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$. The matrix A satisfies its characteristic equation, that is

$$\sum_{j\in S}^{\oplus} \alpha_j \otimes A^{(j)} = \sum_{j\in T}^{\oplus} \alpha_j \otimes A^{(j)}$$

holds for some non-empty, disjoint subsets S and T of the set $\{0, 1, ..., n\}$ and real coefficients α_j, $j \in S \cup T$. Let us multiply this equation by $A^{(k)}$, $k \geq 0$, and then by c^t from the left and by b from the right. Hence we have

$$\sum_{j\in S}^{\oplus} \alpha_j \otimes \left(c^t \otimes A^{(k+j)} \otimes b\right) = \sum_{j\in T}^{\oplus} \alpha_j \otimes \left(c^t \otimes A^{(k+j)} \otimes b\right)$$

or, equivalently

$$\sum_{j\in S}^{\oplus} \alpha_j \otimes g_{k+j} = \sum_{j\in T}^{\oplus} \alpha_j \otimes g_{k+j}$$

for every integer $k \geq 0$. Thus we get for $r \geq n$

$$\sum_{j\in S}^{\oplus} \alpha_j \otimes \gamma_j = \sum_{j\in T}^{\oplus} \alpha_j \otimes \gamma_j,$$

where $\gamma_1, ..., \gamma_r$ denote the columns of H_r. This means that the columns of H_r are linearly dependent. Using Proposition 3.3 we get (see also Gaubert, Butkovič and Cuninghame-Green [16])

Theorem 4.2.
If for some $r > 0$ all maximum assignment solutions of the linear assignment problem with cost matrix H_r have the same parity, then there is no realisation of dimension r or less for the system producing $\{g_j\}_{j=0}^{\infty}$.

As we already mentioned before the question whether all optimal solutions to the assignment problem are of the same parity can be decided in polynomial time, see Theorem 3.4, but the algorithm is involved and is likely to be of high complexity. So the last theorem motivates further research on the linear assignment problem with a symmetric cost matrix. In particular the question remains open, whether the linear assignment problem whose coefficient matrix is a Hankel matrix can be solved in a faster way than in $O(n^3)$ steps. The following proposition shows that we can always achieve a symmetric normal form, if the coefficient matrix of the linear assignment problem is symmetric.

Proposition 4.3. *For a symmetric $n \times n$ matrix A a symmetric normal form can be determined in $O(n^3)$ steps.*

Proof. It is well known that a normal form B of an $n \times n$ matrix A can be found in $O(n^3)$ time by algorithms solving the linear assignment problem. If A is symmetric, then the permutations φ and φ^{-1} yield the same value. Using (10) we get

$$\begin{aligned} 2\sum_{i=1}^{n} a_{i\varphi(i)} &= \sum_{i=1}^{n} a_{i\varphi(i)} + \sum_{i=1}^{n} a_{i\varphi^{-1}(i)} \\ &= 2z + \sum_{i=1}^{n} b_{i\varphi(i)} + \sum_{i=1}^{n} b_{\varphi(i)i} \\ &= 2z + \sum_{i=1}^{n} (b_{i\varphi(i)} + b_{\varphi(i)i}). \end{aligned}$$

Thus,

$$\overline{B} := \frac{1}{2}(B + B^t)$$

is a symmetric normal form of A. □

From the symmetry of the normal form it follows:

Corollary 4.4. *There always exists an optimal solution consisting only of odd cycles (possibly loops) and 2-cycles (edges) in the cyclic representation of φ.*

The Corollary says in particular that every even cycle in the optimal solution can be split into 2-cycles.

5. The eigenvalue problem

It has been shown by Cuninghame-Green [9] that the max-algebraic eigenvalue of a weighted adjacency matrix of a strongly connected digraph is uniquely defined and can be computed as the maximum cycle mean in this digraph. This is true in particular for all matrices with finite entries. Moreover, Cuninghame-Green [11] showed that there is a close connection between the eigenvalue λ of a matrix A and a maxpolynomial which plays the role of the characteristic polynomial in max-algebra. The max-algebraic characteristic polynomial called *characteristic maxpolynomial* of a square matrix A is defined by

$$\chi_A(x) := \mathrm{maper}(A \oplus x \otimes I).$$

In other words, it is the max-algebraic permanent of the matrix

$$\begin{pmatrix} a_{11} \oplus x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} \oplus x & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \oplus x \end{pmatrix}.$$

This means that

$$\chi_A(x) = \delta_0 \oplus \delta_1 \otimes x \oplus \cdots \oplus \delta_{n-1} \otimes x^{(n-1)} \oplus x^{(n)}$$

or written in conventional notation

$$\chi_A(x) = \max\left(\delta_0, \delta_1 + x, \cdots, \delta_{n-1} + (n-1)x, nx\right).$$

Thus, viewed as a function in x, the characteristic maxpolynomial of a matrix A is a piecewise linear, convex function which can be found in $O(n^4)$ steps, see Burkard and Butkovič [2] who improve a previous result by Butkovič and Murfitt [6]. The method in [2] is based on ideas from computational geometry combined with solving assignment problems.

If for some $k \in \{0, ..., n\}$ the inequality

$$\delta_k \otimes x^{(k)} \leq \sum_{i \neq k}^{\oplus} \delta_i \otimes x^{(i)}$$

holds for every real x then the term $\delta_k \otimes x^{(k)}$ is called *inessential*, otherwise it is called *essential*. Note that inessential terms are not needed in order to describe $\chi_A(x)$ as a function of x. Inessential terms will not be found by the method described in [2].

According to Proposition 2.2 the characteristic maxpolynomial admits a representation by linear factors. In particular, Cuninghame-Green [11] showed

Proposition 5.1. *The eigenvalue of a real $n \times n$ matrix A is the largest corner in the representation of the characteristic maxpolynomial by linear factors.*

Example 3. [11] Let

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix}.$$

The characteristic maxpolynomial of this matrix is

$$\begin{aligned}\chi_A(x) &= \text{maper}\begin{pmatrix} 2\oplus x & 1 & 4 \\ 1 & 0\oplus x & 1 \\ 2 & 2 & 1\oplus x \end{pmatrix} \\ &= x^{(3)} \oplus 2 \otimes x^{(2)} \oplus 6 \otimes x \oplus 7 \\ &= (x \oplus 1) \otimes (x \oplus 3)^{(2)}.\end{aligned}$$

The last expression shows that 3 is the largest corner of the characteristic maxpolynomial. Therefore matrix A has the eigenvalue 3. □

Let $A = (a_{ij})$ be an $n \times n$ matrix. Any matrix of the form

$$\begin{pmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \cdots & a_{i_1 i_k} \\ a_{i_2 i_1} & a_{i_2 i_2} & \cdots & a_{i_2 i_k} \\ \vdots & \vdots & & \vdots \\ a_{i_k i_1} & a_{i_k i_2} & \cdots & a_{i_k i_k} \end{pmatrix}$$

with $1 \le i_1 < i_2 < \ldots < i_k \le n$ is called a $(k \times k)$ *principal submatrix.* The *best principal submatrix assignment problem BPSAP(k)* can be stated as follows:
For given k, $1 \le k \le n$*, find a principal submatrix of size k and a permutation* φ *of the set* $\{1, 2, \ldots, k\}$ *such that*

$$\sum_{r=1}^{k} a_{i_r \varphi(i_r)}$$

is minimum (or maximum).

Cuninghame-Green [11] showed that the max-algebraic characteristic polynomial of a matrix A is closely related to the best principal submatrix assignment problem.

Proposition 5.2. *[11] Let* $\chi_A(x) = \delta_0 \oplus \delta_1 \otimes x \oplus \cdots \oplus \delta_{n-1} \otimes x^{(n-1)} \oplus x^{(n)}$ *be the characteristic maxpolynomial of an* $n \times n$ *matrix A. Then the coefficients* δ_k *are given by*

$$\delta_k = \sum_{B \in \mathcal{A}_k}^{\oplus} maper(B), \tag{13}$$

where $\mathcal{A}_k$ *is the set of all principal submatrices of A of size* $(n-k) \times (n-k)$*.*

Obviously, $\delta_0 = \text{maper}(A) = \max_\varphi \sum_{i=1}^n a_{i\varphi(i)}$ and $\delta_{n-1} = \max(a_{11}, a_{22}, \ldots, a_{nn})$.

There is an operations research interpretation – called the *job rotation problem* – of the coefficients of the characteristic maxpolynomial: Suppose that a company with n employees requires these workers to swap their jobs (possibly on a regular basis) in order to avoid exposure to monotonous tasks (for instance manual workers at an assembly line or ride operators in a theme park). It may also be required that to maintain stability of service only a certain number of employees, say k $(k < n)$, actually swap their jobs. With each pair old job – new job a coefficient may be associated expressing

the cost (for instance for an additional training) or the preference of the worker to this particular change. So the aim may be to select k employees and to suggest a plan of the job changes between them so that the sum of the coefficients corresponding to these changes is minimum or maximum. This task leads to finding a $k \times k$ principal submatrix of A for which the assignment problem value is minimum or maximum (the diagonal entries can be set to ∞ or $-\infty$ to avoid an assignment to the same job).
In the next section we discuss the solution of the best principal submatrix assignment problem in greater detail.

6. The best principal submatrix assignment problem

There is no polynomial method known to the authors to solve the best principal submatrix problem in general for a given size k. As we mentioned, however, in the previous section, (BPSAP) can be solved for certain integers k in polynomial time, namely for those k for which δ_{n-k} is an essential term of the characteristic maxpolynomial of the considered matrix A. The graph of the function

$$\chi_A(x) = \max\,(\delta_0, \delta_1 + x, \cdots, \delta_{n-1} + (n-1)x, nx)$$

can be found in the following way (for details, see [2]): For small values of x, the diagonal elements of $A \oplus x \otimes I$ remain unchanged. This means,

$$\delta_0 = \text{maper}(A) = \max_{\varphi \in \mathcal{S}_n} \sum_{1 \le i \le n} a_{i\varphi(i)}.$$

For large values of x we get $\chi_A(x) = nx$. It is easy to see that all possible slopes of the linear parts of $\chi_A(x)$ are $k = 0, 1, 2, ..., n$. The slopes correspond to the number of diagonal elements which are replaced by the current x. In order to find the graph of function $\chi_A(x)$ we intersect the two lines $y = \delta_0$ and $y = nx$ and call the intersection point x_1. In order to determine $\chi_A(x_1)$ we replace all diagonal elements of A which are smaller than x_1 by x_1 and solve the assignment problem with the modified cost matrix. Let the optimum value of the assignment problem be z_1 and let r be the number of diagonal elements in the optimal solution which are equal to x_1. Then $\delta_r = z_1 - rx_1$ and δ_r is the optimum value of the best principal submatrix assignment problem for $k = n - r$.

If $r \ge 2$, we intersect the two lines $y = \delta_0$ and $y = \delta_r + rx$ and get a new value x_2. We proceed as above: we replace all diagonal elements of A which are smaller than x_2 by x_2 and solve the assignment problem with the modified cost matrix. Let the optimum value of the assignment problem be z_2 and let s be the number of diagonal elements which are equal to x_2 and occur in the optimal assignment solution. Then $\delta_s = z_2 - sx_2$ and δ_s is the optimum value of the best principal submatrix assignment problem for $k = n - s$. Since only n different slopes are possible, all linear pieces of the function $\chi_A(x)$ can be found in this way by solving $O(n)$ linear assignment problems. The optimal solutions of the assignment problems whose diagonal entries are modified as above yield also optimal solutions for the corresponding best principal submatrix assignment problems. Since, however, in general not all values k, $0 \le k \le n$, are taken as slopes, i.e. since the characteristic maxpolynomial of matrix A has in general inessential terms, the best principal submatrix assignment problem cannot be solved for all k, $1 \le k \le n$, in this way.

In general, the best principal submatrix assignment problem BPSAP(k) for size k can be described by the following integer program. We modify the linear assignment problem

$$
\begin{aligned}
\max \ & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_{ij} \\
\text{s.t.} \ & \sum_{i=1}^{n} x_{ij} = 1 \quad \text{for all } j = 1, \dots, n, \\
& \sum_{j=1}^{n} x_{ij} = 1 \quad \text{for all } i = 1, \dots, n, \\
& x_{ij} \in \{0, 1\} \quad \text{for all } i, j = 1, \dots, n
\end{aligned}
\tag{14}
$$

by introducing n additional binary variables y_i, $i = 1, 2, ..., n$. Then BPSAP(k) is modelled by the integer program

$$
\begin{aligned}
\max \ & \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_{ij} \\
\text{s.t.} \ & y_j + \sum_{i=1}^{n} x_{ij} = 1 \quad \text{for all } j = 1, \dots, n, \\
& y_i + \sum_{j=1}^{n} x_{ij} = 1 \quad \text{for all } i = 1, \dots, n, \qquad (15) \\
& \sum_{i=1}^{n} y_i = n - k, \\
& x_{ij} \in \{0, 1\} \quad \text{for all } i, j = 1, \dots, n, \\
& y_i \in \{0, 1\} \quad \text{for all } i = 1, \dots, n.
\end{aligned}
\tag{16}
$$

The constraint (16) guarantees that there are k assignment variables x_{ij} equal to 1 and these assignment variables are in rows and columns with the same indices, i.e., in a principal submatrix of size k.

The integer programming formulation given above suggests the following interpretation of the best principal submatrix problem as an *exact weighted matching problem* of the following form: We consider the complete bipartite graph $K_{n,n}$ with n vertices and arcs (i, j'). The weight of arc (i, j') is a_{ij} and can be assumed to be nonnegative. We call all arcs of this graph "blue". Now we introduce n additional "red" arcs (i, i'), $i = 1, 2, .., n$, and give them the weight 0 (this weight does not play any role). Now we ask for a perfect matching in the bipartite multigraph with exactly $n - k$ red arcs and maximum weight. The blue arcs in the perfect matching form the optimal solution of BPSAP(k).

Mulmuley, Vazirani and Vazirani [20] describe a method for solving the exact matching problem (in an arbitrary graph $\bar{G}$) by using the Tutte matrix of $\bar{G}$. Since the Tutte matrix is not defined for multigraphs, we have to get rid of the parallel red and blue arcs (i, i') in order to apply this method. Therefore we construct the following extended graph $\bar{G}$. We replace in G all red arcs (i, i') by a path $(i, i_1, i_2, i_3, i_4, i')$ with new vertices i_1, i_2, i_3 and i_4. The arc (i_2, i_3) is colored red, all other new arcs are colored blue. The weight of all new arcs is set to 0. Now we can show

Lemma 6.1. *Every perfect matching in the multigraph G corresponds uniquely to a perfect matching with the same number of red arcs and the same weight in the extended bipartite graph $\bar{G}$, and vice versa.*

Proof. Let M be a perfect matching of the multigraph G. We extend M to a perfect matching $\bar{M}$ of $\bar{G}$ in the following way: If (i, i') is a red arc in M, we include the arcs (i, i_1), (i_2, i_3) and (i_4, i') in $\bar{M}$. If the red arc (i, i') is not in M, we include the arcs (i_1, i_2) and (i_3, i_4) in $\bar{M}$. It is easy to see that M and $\bar{M}$ are perfect matchings with the same number of red arcs and the same weight.

Conversely, if $\bar{M}$ is an arbitrary perfect matching in $\bar{G}$, then either the arc (i_1, i_2) or the arc (i_2, i_3) are matched, but not both. In the first case the red arc (i, i') of G does not belong to the matching M. In the second case the red arc (i, i') is matched. Thus $\bar{M}$ corresponds uniquely to a perfect matching M of G. □

Due to Mulmuley, Vazirani and Vazirani [20] the exact weighted matching problem in $\bar{G}$ as described above can be solved in randomized polynomial time, provided the weights are polynomially bounded. Thus we get

Theorem 6.2. *If the entries of the matrix A are polynomially bounded, the best principal submatrix assignment problem can for any k, $1 \leq k \leq n$, be solved by a randomized polynomial algorithm.*

7. Conclusion

In the previous sections we outlined the role which assignment problems play in max-algebra. The notion of the determinant as known in usual linear algebra cannot be generalized in a straightforward way to max-algebra. On the contrary, permanents can be generalized. Max-algebraic permanents correspond to linear assignment problems. Many algebraic questions lead to new questions concerning assignment problems. For example, the question whether every optimal assignment solution has the same parity is intimately connected with the regularity of max-algebraic equation systems and the even cycle problem of graphs. Though there exists an involved polynomial algorithm for deciding this problem, still a practically efficient method is not known for this problem. In connection with minimal-dimensional realizations of discrete event systems the problem arises to solve linear assignment problems whose cost matrix is a Hankel matrix. Here the question arises, if this can be done faster than in $O(n^3)$ steps. In connection with eigenvalue problems we introduced the best principal submatrix assignment problem. It is also a suitable tool for modelling the job rotation problem. It is still open, whether the best principal submatrix assignment problem is polynomially solvable or not. In the previous section we showed a close connection between the best principal submatrix assignment problem and the exact (weighted) matching problem, whose complexity status is also still open. Thus a lot has to be done in the future.

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